Janko Bračič

Sets of operators determined by the numerical range

Abstract

Let $\mathscr{H}$ be a complex Hilbert space and $B(\mathscr{H})$ be the Banach algebra of all bounded linear operators on $\mathscr{H}$. The numerical range of $A \in B(\mathscr{H})$ is

$$W(A) = \{\langle Ax, x\rangle; \quad x \in \mathscr{H},\ \|x\| = 1\}.$$

It is well-known that $W(A)$ is a bounded convex subset of complex numbers and that its closure contains the spectrum of A.

For a non-empty bounded set $E \subseteq \mathbb{C}$, let

$$\mathscr{W}_E = \{A \in B(\mathscr{H});\ E \subseteq \overline{W(A)}\}.$$

It is easily seen that this is a non-empty closed set of operators. The case $E = \{0\}$, that is, the set of operators with 0 in the closure of the numerical range, is of a special interest. We will present several results related to the algebraic structure of $\mathscr{W}_{\{0\}}$. For instance, if $\mathscr{H}$ is finite dimensional, then for an operator $A \in \mathscr{W}_{\{0\}}$ there exists a positive semi-definite operator P such that $0 \notin W(PA)$ if and only if 0 is not in the convex hull of the spectrum of A.

Another class of sets determined by numerical ranges are

$$\mathscr{W}^F = \{A \in B(\mathscr{H});\ \overline{W(A)} \subseteq F\},$$

where $F \subseteq \mathbb{C}$ is a given non-empty set. If F is closed, then $\mathscr{W}^F$ is closed in the strong operator topology. Moreover, in this case, $\mathscr{W}^F$ is reflexive in the sense that every operator which is locally in $\mathscr{W}^F$ belongs to $\mathscr{W}^F$. If F is convex, then $\mathscr{W}^F$ is convex, as well. We are able to characterize faces of $\mathscr{W}^F$ in the case when $\mathscr{H}$ is finite dimensional and F is a polyhedron.

The presented results are based on joint papers with Cristina Diogo which have been published during last few years.